Diamagnetic persistent currents for electrons in ballistic billiards subject to a point flux

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We study the persistent current of noninteracting electrons subject to a pointlike magnetic flux in the simply connected chaotic Robnik-Berry quantum billiard and also in an annular analog thereof. For the simply connected billiard, we find a large diamagnetic contribution to the persistent current at small flux, which is independent of the flux and is proportional to the number of electrons (or equivalently the density since we keep the area fixed). The size of this diamagnetic contribution is much larger than mesoscopic fluctuations in the persistent current in the simply connected billiard. Moreover, it can ultimately be traced to the response of the angular-momentum $l=0$ levels (neglected in semiclassical expansions) on the unit disk to a pointlike flux at its center. The same behavior is observed for the annular billiard when the inner radius is much smaller than the outer one. The usual fluctuating persistent current and Anderson-like localization due to boundary scattering are seen when the annulus tends to a one-dimensional ring. We explore the conditions for the observability of this phenomenon.

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I. INTRODUCTION

A resistanceless flow of electrons can occur in mesoscopic systems if the linear size L is less than the phase coherence length L_ϕ. The simplest example of this is a one-dimensional (1D) metallic ring threaded by a magnetic flux Φ. The thermodynamic relation,

$$I=-\frac{\partial F}{\partial \Phi}, \tag{1}$$

defines the persistent current in mks units. At zero temperature, which we will focus on, the free energy F can be replaced by the total ground-state energy E.

Persistent currents were first predicted to occur in superconducting rings.[1–3] It was later realized that persistent currents exist in normal metallic rings as well.[4–6] The phenomenon is understood most easily at zero temperature for a ring of noninteracting electrons, where the electronic wave function extends coherently over the whole ring. If the ring is threaded by a solenoidal flux, all physical properties are periodic in applied magnetic flux with a period of the flux quantum $\Phi_0=h/e$. A nonzero flux splits the degeneracy between clockwise and anticlockwise moving electrons. Upon filling the energy states with electrons, one finds ground states, which have net orbital angular momentum, and net persistent current. Much experimental work has been carried out on ensembles of rings/quantum dots[7,8] in a flux as well as on single metallic[9–13] or semiconductor quantum dots/rings.[14–17] The subject has a long theoretical history as well[18–28] (for a review see Ref. 29).

In this paper we investigate the persistent current of noninteracting electrons in quantum billiards subject to a point flux. Related semiclassical calculations have been carried out in the past for regular (integrable in the absence of flux) (Refs. 30–35) and chaotic billiards.[30,32,33,36] Numerics have previously been performed on these systems as well.[37,38] We carry out calculations on the simply connected chaotic Robnik-Berry billiard[39–43] (also known as a Pascal limaçon), obtained by deforming the boundary of the integrable disk, and on an annular analog, which we call the Robnik-Berry annulus. The ratio of the inner r to the outer radius R of the annulus ($\xi=r/R$) plays an important role in our analysis, and allows us to go continuously between the simply connected chaotic two-dimensional billiard and a (effectively disordered) quasi-one-dimensional ring. We use this billiard for ease of computation. We expect our results to depend on neither the detailed shape of the billiard nor the degree of chaoticity, as will become evident below.

Our main result is that there is a large diamagnetic and flux-independent contribution (for small flux) to the persistent current for $|\Phi|\ll\Phi_0$ in the simply connected billiard, which is proportional to the number of particles and overwhelms the mesoscopic fluctuations, which have been the focus of previous work.[18,19,24–26,30–38] This arises from angular-momentum $l=0$ states in the integrable disk, which respond diamagnetically, with energy increasing *linearly* with the point flux, for small flux. Due to the periodicity of the spectrum under an integer change of point flux, it follows that there is a similar systematic *paramagnetic* contribution for the flux tending to an integer from below. In other words, near an integer multiple n of the flux quantum Φ_0, the energy is proportional to $|\Phi-n\Phi_0|$. This behavior is robust under the deformation of the boundary, which makes the dynamics chaotic. As ξ increases from zero, this contribution to the persistent current persists for typical $\Phi/\Phi_0\simeq1$ but smoothly decreases in magnitude and becomes negligible for $\xi\rightarrow1$. The precise ξ at which the diamagnetic contribution to the persistent current becomes equal to the typical fluctuating paramagnetic contribution depends on the electron density. For $\xi\neq0$ and very tiny flux, the diamagnetic contribution to the persistent current varies linearly with Φ/Φ_0 (see below).

This diamagnetic contribution seems to have been missed in the previous work to the best of our knowledge. The reason is that the semiclassical approximation becomes asymptotically exact as the energy tends to infinity, and in this limit, the spectral density of $l=0$ states vanishes. Thus, $l=0$ states are explicitly disregarded[30,32,33,36] in the semiclassical approach since they do not enclose flux. It has been noted in the past that diffraction effects necessitate an inclusion of $l=0$ states in the sum over periodic orbits on the integrable disk[31] but the connection to persistent currents was not made.

It should be emphasized that since the total persistent current is a sum over the contributions of all levels, the diamagnetic contribution we uncover exists even at very large energies where the levels at the Fermi energy are well approximated by semiclassics.

The robustness of the diamagnetic contribution to the persistent current under deformation can be understood as follows: In the chaotic billiard, each state at a particular energy is roughly a linear combination of states of the regular disk within a Thouless energy ($E_T \simeq \hbar v_F / L$, where L is the linear size of the billiard) of its energy. When the Fermi energy E_F greatly exceeds E_T, the contribution of the occupied states does not change much when the boundary is deformed and chaos is introduced.

The above argument also reveals that the diamagnetic contribution we uncover should not depend on the degree of chaoticity of the billiard since it is descended from the response of the $l=0$ on the regular disk/annulus. Different degrees of chaos can only alter the width of the energy window ($\simeq E_T$) over which the states of the regular disk are spread when the disk/annulus is deformed. Again, for $E_F \gg E_T$, the degree of chaos is seen to be unimportant. The effect we describe is completely generic as long as the states are not localized.

The systematic diamagnetic contribution we uncover appears similar to, but is different from, Landau diamagnetism[44] in a finite system, which is a response to a uniform magnetic field. The primary difference is that the orbital magnetization (proportional to the persistent current) in Landau diamagnetism is proportional to the field itself (because the energy goes quadratically with the field strength), whereas the effect we describe is independent of the flux for small flux in the simply connected Robnik-Berry billiard (because the energy goes linearly with the flux). In the Robnik-Berry annulus with $\xi \to 0$, the energy rises quadratically with the flux for very tiny flux $\Phi \ll \Phi_0 / \log N\xi^{-2}$ but crosses over to the linear behavior characteristic of the simply connected system for larger Φ. Since the flux is pointlike and, in the annular case, nonzero only where the electron wave functions vanish, the entire effect is due to Aharanov-Bohm quantum interference. (The annular case is physically cleaner since there are no diffraction effects associated with the point flux, in contrast to the case of the disk[31]). Since the effect is primarily caused by levels deep below E_F, experimental detection is feasible only through the total magnetization and not by conductance fluctuations that are sensitive to the levels within the Thouless shell (lying within E_T of E_F). Previous samples have been subjected to a uniform field rather than a point flux[9–12,14–16] and anyway the ring samples have ξ too large for this effect to be seen. However, we believe that experiments can be designed to observe this effect.

The plan of the paper is as follows: In Sec. II we describe the method we use to calculate the spectrum, and present analytical expressions and numerical results for persistent current in the disk and simply connected Robnik-Berry billiards. In Sec. III we generalize the method to the annulus and present our results. Conclusions and implications are presented in Sec. IV.

II. SIMPLY CONNECTED ROBNIK-BERRY BILLIARD

We begin by briefly describing the procedure to obtain the energy levels ϵ_k within the billiard, which leads to the persistent current,

$$I = \sum_k I_k, \quad I_k = -\frac{\partial \epsilon_k}{\partial \Phi}. \tag{2}$$

We work with the Robnik-Berry billiard,[40,41] which is obtained from the unit disk by conformal transformation. The original problem of finding energy levels of electron in the domain with complicated boundaries is reduced to a problem where the electron moves in the unit disk in a fictitious potential introduced by the following conformal transformation:

$$w(z) = \frac{z + bz^2 + ce^{i\delta}z^3}{\sqrt{1 + 2b^2 + 3c^2}}, \tag{3}$$

where $w=u+iv$ represents the coordinates in the laboratory coordinate system, and $z=x+iy$ are the conformally transformed coordinates (details are in the Appendix). The parameters b, c, and δ control the shape of the original billiard, and for the values we use, the classical dynamics is mixed but largely chaotic. It is also straightforward to introduce a point flux that penetrates the center of the unit disk[40–43] (after the conformal transformation).

As mentioned in Sec. I, we use this billiard for ease of computation. The results we find are expected to be completely generic and also apply to billiards that are not fully chaotic.

We find 600 energy levels for regular and chaotic billiards for different values of parameter α that controls magnetic flux coming through the billiards. Only the lowest 200 levels are actually used in further calculations since the higher levels become increasingly inaccurate.[42,43] The persistent current is obtained as a numerical derivative of the ground-state energy for a given number of electrons.

For the unit disk billiard the ground-state energy E_G has a nonzero slope as $\alpha \to 0$. Thus there is a persistent current in the system for arbitrarily small magnetic flux (see Fig. 1).

Qualitatively this behavior can be understood as follows. In the absence of magnetic field, energy levels corresponding to orbital quantum numbers $\pm l$ are degenerate. A nonzero Φ lifts the degeneracy, and for small α the two $\pm l$ levels have slopes that are equal in magnitude and opposite in sign. Thus, as long as both are occupied, these levels do not contribute to the net persistent current I. The only nonzero contribution comes from levels with $l=0$.

For the unit disk the expression for the persistent current can be derived analytically for small values of magnetic flux. At zero temperature the persistent current due to kth level is $I_k = -\partial\epsilon_k / \partial\Phi$ (ϵ_k is a dimensionless energy, and I_k is persistent current divided by the energy unit $\hbar^2/2mR^2$; see the Appendix for notations). For the unit disk, the energy levels are found from the quantization condition,

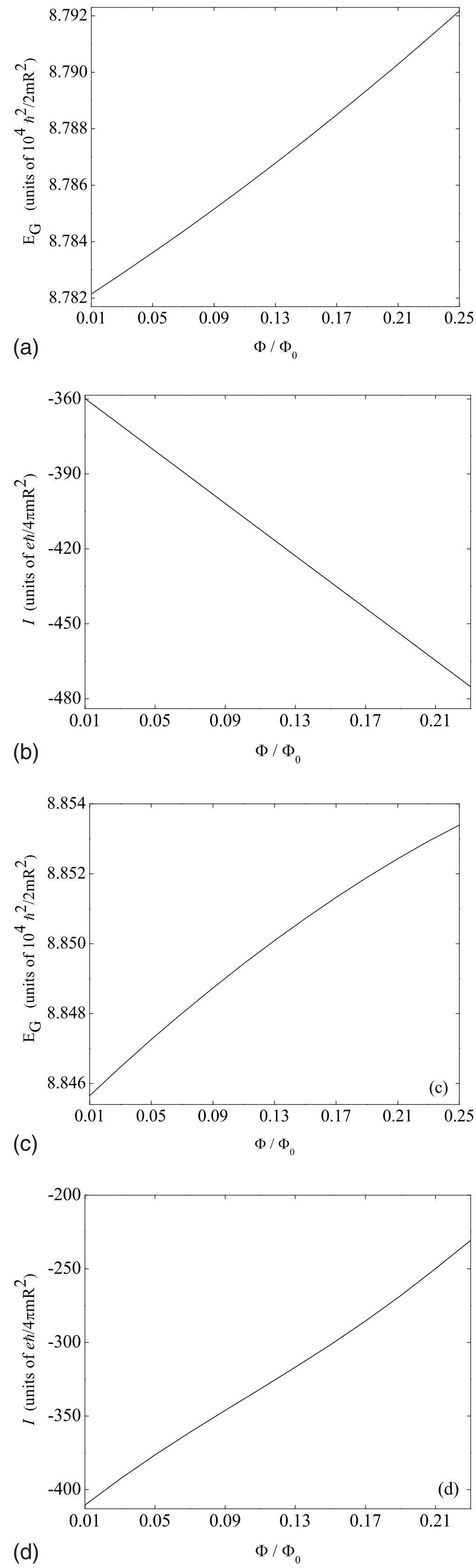

FIG. 1. Ground-state energy E_G in units of $10^4 \times \frac{\hbar^2}{2mR^2}$ and persistent current I in units of $\frac{e\hbar}{4\pi mR^2}$ as a function of dimensionless flux for the regular disk (panels a and b), and the simply connected chaotic billiard (panels c and d). The results are for 200 particles.

$$J_{|l-\alpha|}[\gamma_n(|l-\alpha|)] = 0, \quad \epsilon_k = \gamma_n^2(|l-\alpha|). \tag{4}$$

Then from Eq. (2), the persistent current caused by kth level is

$$I_k = -\frac{2e}{h}\gamma_n(|l_k-\alpha|)\frac{\partial\gamma_n(|l_k-\alpha|)}{\partial\alpha}. \tag{5}$$

To find $\partial\gamma/\partial\alpha$ we differentiate Eq. (4) as

$$\frac{\partial J_\nu(\gamma)}{\partial\alpha} = \frac{\partial J_\nu(\gamma)}{\partial\nu}\frac{\partial\nu}{\partial\alpha} + \frac{\partial J_\nu(\gamma)}{\partial\gamma}\frac{\partial\gamma}{\partial\alpha} = 0. \tag{6}$$

For $l=0$ levels, $\nu=|l-\alpha|=\alpha$. In $\alpha\to 0$ limit, for the derivatives of Bessel function, one gets

$$\left.\frac{\partial J_\nu(\gamma)}{\partial\nu}\right|_{\nu=0} = \frac{\pi}{2}N_0(\gamma),$$

$$\left.\frac{\partial J_\nu(\gamma)}{\partial\gamma}\right|_{\nu=0} = -J_1(\gamma). \tag{7}$$

Combining Eqs. (6) and (7) and using relation that, for $\gamma \gg 1$, Bessel function $N_0(\gamma)\approx J_1(\gamma)$ [this approximation works well already for the first root of Eq. (4)], we find $\frac{\partial\gamma}{\partial\alpha}|_{\alpha=0} = \frac{\pi}{2}$, which leads to

$$I = -\frac{\pi e}{h}\sum_n \gamma_n(0), \tag{8}$$

where summation is over the levels with orbital quantum number $l=0$.

For large argument values (which is the same as large energies), the quantization condition [Eq. (4)] for the unit disk becomes $\cos(\gamma_n - \pi\alpha/2 - \pi/4)=0$, with roots

$$\gamma_n = \pi\alpha/2 + \pi/4 + \pi(2n+1)/2. \tag{9}$$

With the energy being measured in $\hbar^2/2mR^2$ units, the Fermi wave vector is $k_F=\gamma_{\max}\approx\pi n_{\max}$, where $n_{\max}$ denotes the largest $l=0$ level. With disk area equal to π ($R=1$), the number of particles in the system is $N=(\pi n_{\max}/2)^2$. This allows us to find the dependence of the persistent current on number of particles in the system in $\alpha\to 0$ limit,

$$I = -\frac{e\pi}{h}\sum_n\left(\frac{3\pi}{4}+\pi n\right) \approx -\frac{e\pi^2}{2h}n_{\max}^2 = -\frac{e}{h}2N, \tag{10}$$

where we neglected a subleading term proportional to $n_{\max}$. We remind the reader that the physical persistent current is the expression in formula (10) multiplied by energy unit $\hbar^2/2mR^2$.

In Fig. 2 the persistent current I is plotted against the number of particles N for magnetic flux $\alpha=0.01$. The behavior of the current is consistent with Eq. (10). That is, for small magnetic flux, it is proportional to $2N$. For the regular disk [Fig. 2(a)], the persistent current is a set of consecutive steps. Each step appears when the next $l=0$ level is added to the system. The length of the steps is equal to the number of $l\neq 0$ levels between two adjacent levels with zero orbital quantum number. As one particle is added to the $l\neq 0$ level, it results in persistent current jump. The next level corre-

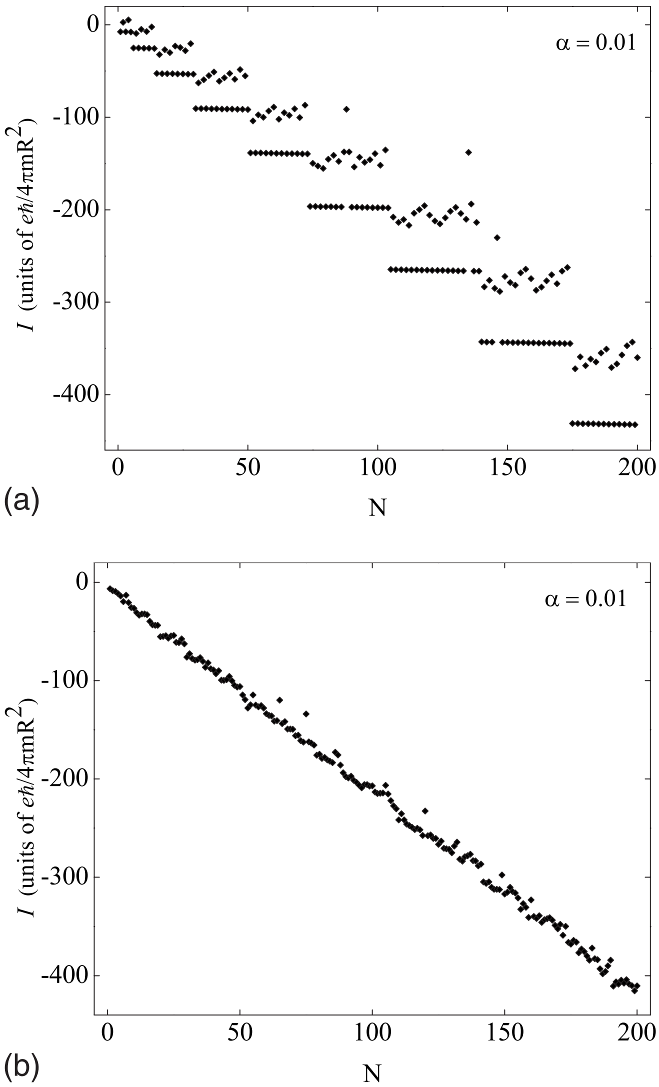

FIG. 2. Persistent current I vs number of particles N for (a) regular disk and (b) chaotic disk for the value of reduced magnetic flux α=0.01.

sponding to $-l$ has opposite slope, and once it is occupied, cancels the contribution of the previous l level to the net persistent current. This explains the noise above each step in Fig. 2(a). The noise appears on the level of previous step since the slopes of the current l=0 level and all nearby $l \neq 0$ levels do not differ much. In addition, each step has a small inclination, which is due to the fact that the $l \neq 0$ levels do not cancel each other exactly when $\Phi \neq 0$. For larger magnetic flux the steps become more inclined.

To see that levels with $l \neq 0$ do not contribute to the persistent current at weak magnetic flux, we simply note that derivative of γ,

$$\left.\frac{\partial \gamma_n}{\partial \alpha}\right|_{\alpha=0} = - \frac{\partial J_\nu(\gamma_n)}{\partial \nu}\frac{\partial \nu}{\partial \alpha} \Bigg/ \left.\frac{\partial J_\nu(\gamma_n)}{\partial \gamma_n}\right|_{\alpha=0}, \qquad (11)$$

is an odd function of l. In the $\alpha \to 0$ limit, the root $\gamma_n(\nu)$ and the derivatives of $J_\nu(\gamma_n)$ in Eq. (11) are even functions of l, and $\partial \nu / \partial \alpha$ is odd. As a result, the whole expression is an odd function of l, which proves the cancellation of $\pm l$ levels in Eq. (5).

For the chaotic simply connected Robnik-Berry billiard, each eigenstate is a superposition of all l states of regular disk [see Eq. (A5)], mostly within a Thouless shell of its energy. Assuming that states with $\pm l$ enter this superposition with equal probability over the ensemble due to the chaotic nature of motion, one can conclude that the ensemble-averaged contribution of these levels to the net current is zero. Thus, as seen in Fig. 2(b), the mesoscopic fluctuations due to the $l \neq 0$ levels are overwhelmed by the diamagnetic contribution linear in N for small α.

III. ROBNIK-BERRY ANNULUS

Now we turn our attention to annular billiard. Here there is an additional parameter ξ, which is the ratio of the inner radius r to the outer radius R of the regular annulus in the (conformally transformed) z plane. By varying ξ we are able to smoothly go from the simply connected Robnik-Berry billiard to a (effectively) disordered ring in the limit $\xi \to 1^-$.

First consider the disk limit $\xi \to 0$. We can derive an analytical expression for I when ξ is small enough that $\xi\gamma_n \ll 1$ and $\gamma_n \gg 1$. For a regular annulus with R=1, energy quantization follows from the Dirichlet boundary condition,

$$J_\nu(\gamma_n)N_\nu(\gamma_n\xi) - J_\nu(\gamma_n\xi)N_\nu(\gamma_n) = 0. \qquad (12)$$

Here n numerates root at fixed angular momentum ν. We use the large and small argument expansions for Bessel functions to obtain, for the l=0 levels,

$$\cot\left(\gamma_n - \frac{\pi\alpha}{2} - \frac{\pi}{2}\right) = \frac{\dfrac{1}{\Gamma(1+\alpha)}\left(\dfrac{\gamma_n\xi}{2}\right)^{\alpha}}{\dfrac{\cot(\alpha\pi)}{\Gamma(1+\alpha)}\left(\dfrac{\gamma_n\xi}{2}\right)^{\alpha} - \dfrac{\Gamma(\alpha)}{\pi}\left(\dfrac{\gamma_n\xi}{2}\right)^{-\alpha}}. \qquad (13)$$

We express the roots for the annulus as a small deviation from the roots for the disk, which we denote as $\gamma_n^{(d)}$; $\gamma_n = \gamma_n^{(d)} + \delta\gamma_n$ with $\gamma_n^{(d)} = \alpha\pi/2 + \pi/4 + \pi(2n+1)/2$. Approximating $\cot(\alpha\pi)$ by $1/(\alpha\pi)$, for small values of $\delta\gamma_n$, we find

$$\delta\gamma_n = -\frac{\alpha\pi}{2}\left[1 + \coth\left(\alpha \ln\frac{\gamma_n\xi}{2}\right)\right]. \qquad (14)$$

In Eq. (14), for small α, γ_n under logarithm can be safely replaced by its value for the disk $\gamma_n^{(d)}$. Then roots for the annulus are

$$\gamma_n = \frac{\pi}{4} + \frac{\pi}{2}(2n+1) - \frac{\pi}{2}\alpha \coth\left[\alpha \ln\frac{\gamma_n^{(d)}\xi}{2}\right]. \qquad (15)$$

One can now take various limits of Eq. (15). To recover Eq. (9) for the disk roots, we keep magnetic flux α fixed and take the limit $\xi \to 0$. As one can see from Eq. (15), convergence to the disk limit is slow due to the logarithm, and occurs only for $\alpha \gg 1/\log[\gamma_n^{(d)}\xi/2]$.

Another limit of interest is to keep ξ fixed and obtain behavior of roots γ_n for small α. For small α

$\ll 1/\log[\gamma_n^{(d)}\xi/2]$, the roots γ_n with $l=0$ vanish quadratically with α. Expanding the coth function in Eq. (15), we obtain

$$\gamma_n = \frac{\pi}{4} + \frac{\pi}{2}(2n+1) - \frac{\pi}{2}\left[1 + \frac{\alpha^2}{3}\ln^2\frac{\gamma_n^{(d)}\xi}{2}\right]\ln^{-1}\frac{\gamma_n^{(d)}\xi}{2}, \tag{16}$$

which leads to the persistent current:

$$I \approx \frac{2\pi e\alpha}{3h}\sum_n \left\{\left[\frac{\pi}{4} + \frac{\pi}{2}(2n+1)\right]\ln\frac{\gamma_n^{(d)}\xi}{2} - \frac{\pi}{2}\right\}. \tag{17}$$

Rough estimation of this sum with the help of the Euler-MacLaurin formula gives

$$I \approx \frac{\pi^2 e\alpha}{3h}n_{\max}^2 \ln\frac{n_{\max}\xi\pi}{2\sqrt{e}}, \tag{18}$$

where we kept only terms proportional to $n_{\max}^2$, and $e=2.718\,28\ldots$ inside the logarithm denotes Euler's number and not the electronic charge.

Using the relation $N=(\pi n_{\max}/2)^2$ (for small values of ξ, the density of states for the annulus and the disk are practically the same), the persistent current becomes, for small $\alpha \ll 1/\log[\gamma_n^{(d)}\xi/2]$,

$$I = I^{(d)}\frac{\alpha}{3}\left|\ln\frac{N\xi^2}{\mathrm{e}}\right|, \quad I^{(d)} = -\frac{e}{h}2N. \tag{19}$$

To approach the limit of a one-dimensional ring, where $\gamma_n \gg 1$ and $\gamma_n\xi \gg 1$, we return to quantization condition [Eq. (12)] and use the following large argument expansion for Bessel functions:

$$J_\nu(z) \approx \sqrt{\frac{2}{\pi z}}\left[\cos\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) - \sin\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)\frac{\nu^2 - 1/4}{2z}\right],$$

$$N_\nu(z) \approx \sqrt{\frac{2}{\pi z}}\left[\sin\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) + \cos\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)\frac{\nu^2 - 1/4}{2z}\right]. \tag{20}$$

We use formulas (20) and quantization condition [Eq. (12)] to get an equation for roots:

$$\sin(\gamma_n\sigma) - \cos(\gamma_n\sigma)\frac{\nu^2 - 1/4}{2\gamma_n\xi}\sigma = 0, \tag{21}$$

where we ignore the term that is proportional to $1/\gamma_n^2$. The quantity $\sigma=1-\xi$ is assumed to be much less than unity. For sufficiently small σ, one can drop the second term in Eq. (21) and get $\gamma_n\sigma=\pi n$. To find corrections to this expression, we assume that $\gamma_n\sigma=\pi n+\eta$ with $\eta=\frac{\nu^2-1/4}{2\pi n}\sigma^2 \ll 1$, and plug it in Eq. (21) to obtain the solutions of quantization condition

$$\gamma_n = \frac{\pi n}{\sigma} + \frac{\nu^2 - 1/4}{2\pi n}\sigma, \quad \nu = |l - \alpha|. \tag{22}$$

The energy spectrum for the annulus in this limit is

$$\epsilon_{n,l} = \gamma_n^2 \approx \left(\frac{\pi n}{\sigma}\right)^2 + (\nu^2 - 1/4). \tag{23}$$

The first term in Eq. (23) denotes the radial kinetic energy and diverges in $\sigma \to 0$ limit. This divergence can be absorbed into the chemical potential for the $n=1$ radial state. The difference between energy levels with radial quantum numbers n is of the order $n(\pi/\sigma)^2$. For $\sigma\to 0 \Rightarrow \xi \to 1$, one can assume that all the levels of interest have the radial quantum number $n=1$ and are labeled only by orbital quantum number l. Since our diamagnetic persistent current arises from a large number $\propto\sqrt{N}$ of $l=0$ levels, it is clear that it vanishes in the limit of a ring.

It is straightforward to show that for a regular annulus the contributions of $\pm l$ levels also cancel each other for small values of α. However, levels with $l=0$ have zero slope when $\alpha\to 0$. To show this one takes the derivative of quantization condition [Eq. (12)]:

$$\dot{J}_\nu(\gamma_n\xi)N_\nu(\gamma_n) + J_\nu(\gamma_n\xi)\dot{N}_\nu(\gamma_n) - \dot{J}_\nu(\gamma_n)N_\nu(\gamma_n\xi) - J_\nu(\gamma_n)\dot{N}_\nu(\gamma_n\xi) + \frac{\partial\gamma_n}{\partial\alpha}[\xi J_\nu'(\gamma_n\xi)N_\nu(\gamma_n) + J_\nu(\gamma_n\xi)N_\nu'(\gamma_n) - J_\nu'(\gamma_n)N_\nu(\gamma_n\xi) - \xi J_\nu(\gamma_n)N_\nu'(\gamma_n\xi)] = 0, \tag{24}$$

where $\dot{A}_\nu(z)=\partial A_\nu(z)/\partial\nu$, and $A_\nu'(z)=\partial A_\nu(z)/\partial z$. When $\alpha\to 0$, derivatives of Bessel functions become $\dot{J}_\nu(z)=\pi N_0(z)/2$, $\dot{N}_\nu(z)=-\pi J_0(z)/2$, $J_\nu'(z)=-J_1(z)$, and $N_\nu'(z)=-N_1(z)$. Then all terms outside square brackets in Eq. (24) cancel each other. The expression inside brackets in general has a nonzero value, which means $\partial\gamma_n/\partial\alpha=0$.

In Fig. 3 the persistent current in the annular billiard is depicted for different values of the aspect ratio ξ. To facilitate the comparison between different values of ξ, we keep the area of the annulus the same, thus keeping the average density of states the same. For a regular annulus [Fig. 3(a)] for small values of flux, the current is a linear function of α. As ξ gets smaller, the diamagnetic contribution to the persistent current increases. This behavior is consistent with Eq. (19) that shows linear dependence on α and slow growth as $\xi\to 0$.

In the regular annulus, for ξ close to unity, the behavior of the persistent current is close, but not identical, to that of a 1D ring. Even for $\xi=0.9$ there exist several states with $l=0$, which means that our billiard is not purely a 1D ring. The effect of these states on the persistent current is not entirely trivial. For a fixed number of particles in the system, as ξ changes, the number of $l=0$ levels also changes. As the next $l=0$ level is added (or expelled), the current experiences a jump. The magnitude of this jump is large enough for small α to make the current positive. For larger α the current remains diamagnetic.

In the distorted annulus [Fig. 3(b)], the persistent current is a linear function of α for small α. For larger magnetic flux, one observes nonlinear behavior that can be attributed to level repulsion in the chaotic billiard.

The dependence of the persistent current in the annulus on the number of particles N at fixed α is similar to that in the simply connected billiard. At small ξ the persistent current in

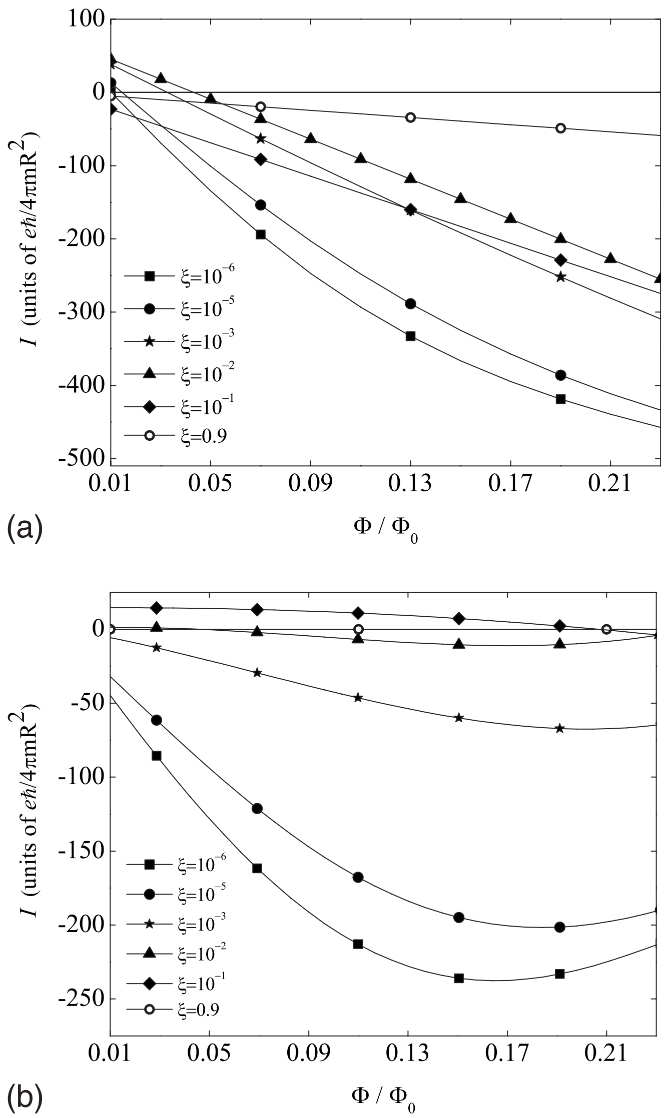

FIG. 3. Persistent current I vs reduced magnetic flux α for several values of ξ for $N=200$ particles. (a) represents current for regular annulus normalized to the same density of states (same area) for different values of ξ. The current for the chaotic annulus is depicted in (b).

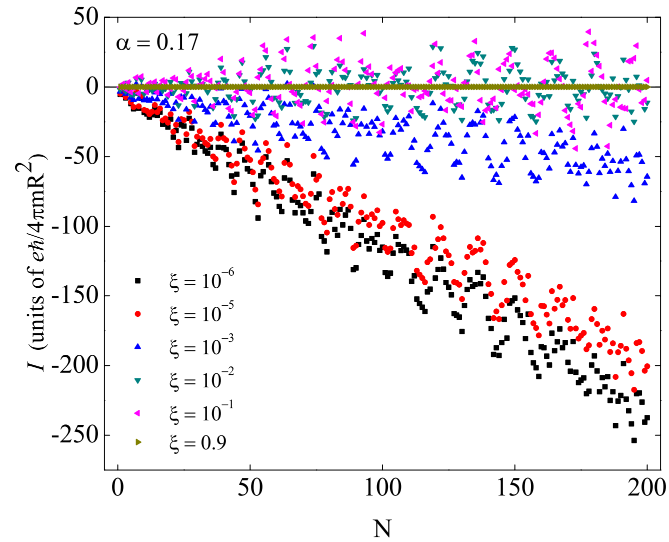

FIG. 4. (Color online) Persistent current I in distorted annulus vs number of particles N for several values of ξ. Magnetic flux $\alpha=0.17$.

the regular annulus is a staircaselike function. For the distorted annulus the numerics are scattered around a straight line (see Fig. 4).

For larger ξ the magnitude of diamagnetic contribution to the persistent current decreases, and the numerics are dominated by mesoscopic fluctuations. When $\xi \to 1$, the persistent current becomes negligible [see Fig. 5(a)] for low occupations. We believe this is a manifestation of Anderson localization due to the boundary scattering. At high energies, when the localization length exceeds the circumference of the annulus, extended states reappear and can carry the persistent current. In Fig. 5 we plot the persistent current for two different sets of parameters controlling the shape of annulus. For a large distortion [Fig. 5(a)], the current is nonzero only for high energy states beyond $N=110$.

In Fig. 5(b) the parameters b, c, and δ are chosen to make the annulus less distorted, and we see that the threshold for extended states moves to lower energy (about $N=40$). The positive branch of persistent current in Fig. 5(b) can be explained as follows. For the relevant values of b, c and δ, the chaotic annulus is almost regular. For small distortions we can loosely speak in terms of states with definite values of orbital momentum (let us say that dominant contribution comes from the state with orbital momentum l). Then the positive branch signifies the occupation of a $l>0$ state while the addition of another particle into the $-l$ state brings the total magnetization back to near zero.

IV. CONCLUSIONS, CAVEATS, AND OPEN QUESTIONS

We have investigated the behavior of chaotic simply connected and annular billiards penetrated by a pointlike flux. The annular billiards are characterized by a dimensionless aspect ratio $\xi=r/R$, the ratio of the inner (r) to the outer radius (R). Note that in the annular billiards, the flux exists in a region where the electrons cannot penetrate, and the effects of the flux on the electrons are purely Aharanov-Bohm quantum interference effects. We emphasize this point since in a simply connected billiard, there are diffraction effects associated with a point flux as well,[31] and we want to separate those from quantum interference effects.

Our main result is that there is a systematic diamagnetic contribution to the persistent current, which can be traced back to the flux response of the $l=0$ levels of a regular unit disk (or annulus). Even though the number of such $l=0$ levels is submacroscopic ($\propto\sqrt{N}$, where N is the number of electrons), the contribution to the persistent current due to these levels is proportional to N and is independent of the flux for small flux in simply connected billiards. Moreover, it can overwhelm the fluctuating mesoscopic contribution[18–26,30,32,33,36] from the states in the Thouless shell ($|E-E_F|\leq E_T$). This effect is quite distinct from Landau diamagnetism.[44] Near an integer multiple of the flux quan-

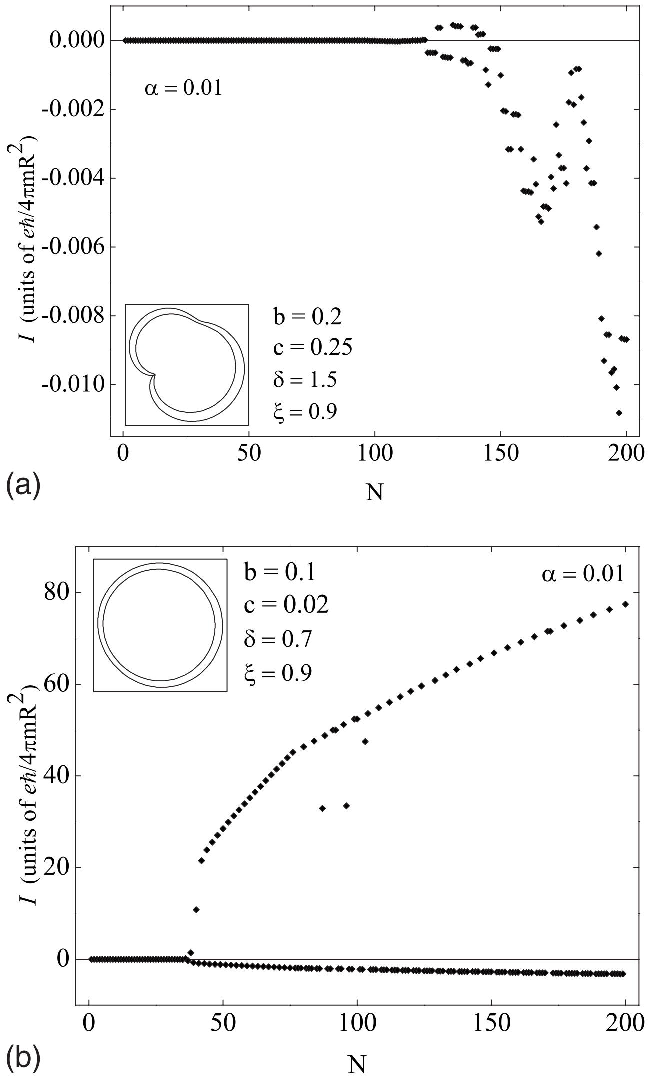

FIG. 5. Persistent current I vs number of particles N for distorted annulus. Parameters b, c, and δ control the shape of billiard.

tum $\Phi \simeq n\Phi_0$, the flux dependence of the energy of the simply connected billiard is proportional to $|\Phi - n\Phi_0|$, which also implies a large *paramagnetism* as Φ approaches $n\Phi_0$ from below.

The diamagnetic contribution to the persistent current from $l=0$ states seems to have been missed in the previous work, using the semiclassical sum over periodic orbits.[30,32,33,36] This is understandable since the semiclassical approach becomes exact only as $E \to \infty$, and in this limit, the $l=0$ states have vanishing spectral density $\rho_{l=0}(E) \simeq 1/\sqrt{E}$. However, we emphasize that the total persistent current contains the sum over all levels and will indeed behave diamagnetically at small flux (in the simply connected billiard), as we have described. The diamagnetic contribution we uncover is also independent of the degree of chaoticity as long as states are not localized. This is clear from the fact that each exact eigenstate of energy E of the deformed billiard is roughly a superposition of states of the regular billiard within a Thouless energy E_T of E. For $E_F \gg E_T$, the diamagnetic contribution is independent of E_T, and hence on the degree of chaoticity.

For very tiny ξ, the annular Robnik-Berry billiard behaves much like the simply connected one for most values of the dimensionless flux $\alpha = \Phi/\Phi_0 \gg 1/\log N\xi^{-2}$, with a diamagnetic contribution to the persistent current that is proportional to the electron density. However, convergence to the $\xi=0$ limit is logarithmically slow, and the limits $\alpha \to 0$ and $\xi \to 0$ do not commute. As the aspect ratio ξ increases and the annulus tends to a one-dimensional ring, the systematic diamagnetic contribution diminishes to zero. For ξ close to one, we also see Anderson localization in the distorted annular billiards, wherein the persistent currents are negligible below a certain energy (presumably because the localization length for these levels is smaller than the circumference), and become nonzero only beyond a threshold energy.

While we can obtain analytical estimates for the limits $\xi \to 0$ and $\xi \to 1$, it is difficult to make analytical progress for generic values of ξ (not close to zero or one). However, one can easily verify from the asymptotic expansions that for generic ξ the diamagnetic contribution to the persistent current for $\Phi \ll \Phi_0$ goes as

$$I_{\text{dia}} \simeq -\frac{\hbar^2}{\pi m r R}\alpha\sqrt{\frac{2N(R-r)}{(R+r)}}, \qquad (25)$$

where r and R are the inner and outer radii, respectively. This should be compared to the typical fluctuating persistent current for interacting particles,[20–23] which behaves as

$$I_{\text{fluc}} \simeq \frac{E_T}{\Phi_0} \simeq \frac{\hbar^2}{mR\Phi_0}\sqrt{\frac{N}{R(R-r)}}. \qquad (26)$$

It can be seen that the ratio of the systematic diamagnetic persistent contribution to the fluctuating contribution is roughly

$$\frac{|I_{\text{dia}}|}{|I_{\text{fluc}}|} \simeq \frac{(R-r)}{r}\frac{\Phi}{\Phi_0}. \qquad (27)$$

Previous ring samples[9–15,17] have $(R-r) \ll r$. They are also subject to a uniform magnetic field rather than a point flux. Despite this, a systematic diamagnetic contribution at low flux has been detected in recent experiments.[13,17] However, the experiments are carried out at finite frequency, and the effects of attractive pair interactions[45,46] (see below) or non-equilibrium noise[47] cannot be ruled out.

In order to detect this effect unambiguously, one must work with a material that has no superconductivity at any temperature to rule out attractive pair interactions. It is also clear that $\frac{R-r}{r}$ needs to be made as large as possible in order to render this effect easily observable. Care must be taken that there is no magnetic flux in the region where the electron wave functions are nonzero in order to maintain the pure Aharanov-Bohm quantum interference nature of this effect.

Let us now mention some caveats about our work. We have taken only a few ($\simeq 200$) levels into account whereas most experimental samples have a hugely greater number of levels. However, the physics of the diamagnetic contribution to the persistent current for a particular level concerns only whether that level has $l=0$ or not, and is independent of its relative position in the spectrum. We expect our conclusions to hold for arbitrary densities.

We have considered a pointlike flux, which is unachievable in practice. For the annular billiard, all one needs to ensure is that the flux is nonzero only in the central hole of the annulus and is zero in regions where the electron density is nonzero, thus avoiding possible contamination from diffraction effects.[31] By gauge invariance, such a situation will be equivalent to the one we study.

We have also ignored the effect of interelectron interactions. For weak *repulsive* interactions,[48–51] we expect interactions to modify the effect only slightly because it comes primarily from occupied levels deep within the Fermi sea, which are Pauli blocked from responding to the interactions. However, for strong repulsive interactions,[52–54] significant corrections to the persistent current[55] from electrons in the Thouless shell cannot be ruled out. If the interactions are weak but *attractive*,[20–23] the low-energy fluctuations of Cooper pairs become very important,[45,46] and can produce additional large diamagnetic contributions at low fields.

Similarly, although we have concentrated on the zero-temperature behavior, we expect this effect to persist in quite high temperatures since most of the $l=0$ levels involved lie deep within the Fermi sea.

Finally, it would be interesting to investigate the effects of static disorder within a chaotic billiard, which would induce the system to cross over from a ballistic/chaotic to a disordered (diffusive) system. We hope to address this and other issues in future work.

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APPENDIX: NUMERICS FOR ENERGY LEVELS

The idea of this method is as follows.[40–43] In the original uv domain the Schrödinger equation is

$$\frac{1}{2m}[-i\hbar\nabla - q\mathbf{A}(u,v)]^2\Psi(u,v) = E\Psi(u,v), \quad q=-e<0. \tag{A1}$$

To keep the dynamics of the electron unchanged, it is assumed that the magnetic field exists only at the origin of uv plane inside the billiard. This requires that the vector potential satisfies the condition $\nabla\times\mathbf{A}(\mathbf{r})=\mathbf{n}\Phi\delta(\mathbf{r})$, where $\mathbf{n}$ is a unit vector perpendicular to the plane of the billiard.

The billiard is threaded by a single magnetic-flux tube. The strength of the flux is $\Phi=\alpha\Phi_0$, where $\Phi_0=h/e$ is the magnetic-flux quantum.

If the vector potential has the form

$$\mathbf{A}(u,v) = \frac{\alpha}{2\pi}\Phi_0\left(\frac{\partial f}{\partial v},-\frac{\partial f}{\partial u},0\right), \quad f=\frac{1}{2}\ln|z|^2, \tag{A2}$$

then with the help of the conformal transformation

$$w(z) = \frac{z+bz^2+ce^{i\delta}z^3}{\sqrt{1+2b^2+3c^2}}, \quad w=u+iv, \quad z=x+iy, \tag{A3}$$

the Schrödinger equation in polar coordinates becomes

$$\nabla^2_{r,\theta}\Psi(r,\theta) - \frac{i2\alpha}{r^2}\partial_\theta\Psi(r,\theta) - \frac{\alpha^2}{r^2}\Psi(r,\theta) + \epsilon|w'(re^{i\theta})|^2\Psi(r,\theta) = 0. \tag{A4}$$

Here the energy ϵ is measured in units of $\hbar^2/2mR^2$ and the distance is in units of R, where R is the radius of the disk in the xy plane. Also, the coefficients b, c, and δ in Eq. (A3) are real parameters selected in such a way that the transformation $w(z)$ is nonsingular ($|w'(z)|>0$) for all values of z inside the disk. The transformation $w(z)$ is a cubic polynomial normalized to preserve the area of the billiard and leave the average density of states invariant. Equation (A4) should be accompanied by the Dirichlet boundary condition.

To find the energy spectrum, one expands $\Psi(r,\theta)$ in Eq. (A4) in terms of the eigenstates $\phi_{l,n}(r,\theta)$ of a free electron ($w=0$) inside the unit disk:

$$\Psi_p(r,\theta) = N_p\sum_{j=1}^{\infty}\frac{c_j^{(p)}}{\gamma_j}\phi_j(r,\theta). \tag{A5}$$

The index $j=(l,n)$ enumerates levels in ascending order. The normalized function $\phi_{l,n}(r,\theta)$ is

$$\phi_{l,n}(r,\theta) = \frac{J_{|l-\alpha|}[\gamma_n(|l-\alpha|)r]e^{il\theta}}{\sqrt{\pi}J'_{|l-\alpha|}[\gamma_n(|l-\alpha|)]}, \tag{A6}$$

where $J_\nu(r)$ is the Bessel function of the first kind, $\gamma_{l,n}$ is the nth root of $J_\nu(r)$, and l is an orbital quantum number. The coefficients of the expansion in Eq. (A5) are chosen to make the matrix M_{ij} (below) Hermitian.

Plugging the expansion [Eq. (A5)] into Eq. (A4) after simplification, one gets the matrix equation for the eigenvalue problem:

$$M_{ij}c_j^{(p)} = \frac{1}{\epsilon_p}c_i^{(p)}, \tag{A7}$$

where the matrix M is

$$M_{ij} = \left\{\frac{\delta_{ij}}{\gamma_i\gamma_j} + \delta_{l_i,l_j-2}6ce^{-i\delta}I_{ij}^{(2)} + \delta_{l_i,l_j-1}[4bI_{ij}^{(1)} + 12bce^{-i\delta}I_{ij}^{(3)}] + \delta_{l_i,l_j}[8b^2I_{ij}^{(2)} + 18c^2I_{ij}^{(4)}] + \delta_{l_i,l_j+1}[4bI_{ij}^{(1)} + 12bce^{i\delta}I_{ij}^{(3)}] + \delta_{l_i,l_j+2}6ce^{i\delta}I_{ij}^{(2)}\right\}\Big/(1+2b^2+3c^2). \tag{A8}$$

The integrals $I_{ij}^{(h)}$ have the form

$$I_{ij}^{(h)} = \frac{\int_0^1 drr^{h+1} J_{\nu_i}(\gamma_i r) J_{\nu_j}(\gamma_j r)}{\gamma_i \gamma_j J'_{\nu_i}(\gamma_i) J'_{\nu_j}(\gamma_j)}. \tag{A9}$$

Along with the simply connected domain (irregular disk), we consider the irregular annulus. Using a similar conformal transformation, we map the annulus with irregular boundaries from the uv plane onto the regular annulus in the xy plane with inner radius ξ and outer radius $R=1$. The area-preserving conformal transformation is

$$w(z) = \frac{z + bz^2 + ce^{i\delta} z^3}{\sqrt{1 + 2b^2(1+\xi^2) + 3c^2(1+\xi^2+\xi^4)}}. \tag{A10}$$

For this kind of billiard the expansion of $\Psi(r,\theta)$ is in terms of eigenstates $\phi_j(r,\theta)$ for the regular annulus

$$\phi_{l,n}(r,\theta) = \frac{\left\{ J_{|l-\alpha|}[\gamma_n(|l-\alpha|)r] - \frac{J_{|l-\alpha|}[\gamma_n(|l-\alpha|)\xi]}{N_{|l-\alpha|}[\gamma_n(|l-\alpha|)\xi]} N_{|l-\alpha|}[\gamma_n(|l-\alpha|)r] \right\} e^{il\theta}}{\sqrt{2\pi}\sqrt{\int_\xi^1 drr \left\{ J_{|l-\alpha|}[\gamma_n(|l-\alpha|)r] - \frac{J_{|l-\alpha|}[\gamma_n(|l-\alpha|)\xi]}{N_{|l-\alpha|}[\gamma_n(|l-\alpha|)\xi]} N_{|l-\alpha|}[\gamma_n(|l-\alpha|)r] \right\}^2}}. \tag{A11}$$

The counterpart of the matrix M for the annulus is

$$M_{ij} = \left\{ \frac{\delta_{ij}}{\gamma_i\gamma_j} + \delta_{l_i,l_j-2} 3ce^{-i\delta} I_{ij}^{(2)} + \delta_{l_i,l_j-1}[2bI_{ij}^{(1)} + 6bce^{-i\delta} I_{ij}^{(3)}] + \delta_{l_i,l_j}[4b^2 I_{ij}^{(2)} + 9c^2 I_{ij}^{(4)}] + \delta_{l_i,l_j+1}[2bI_{ij}^{(1)} + 6bce^{i\delta} I_{ij}^{(3)}] + \delta_{l_i,l_j+2} 3ce^{i\delta} I_{ij}^{(2)} \right\} \Big/ (1 + 2b^2(1+\xi^2) + 3c^2(1+\xi^2+\xi^4)), \tag{A12}$$

where the integrals $I_{ij}^{(h)}$ are defined as

$$I_{ij}^{(h)} = \int_\xi^1 drr^{h+1} \frac{\tilde{\phi}_i(r)\tilde{\phi}_j(r)}{\gamma_i\gamma_j}, \quad \tilde{\phi}_i(r) = \sqrt{2\pi} e^{-il\theta} \phi_i(r,\theta). \tag{A13}$$

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